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Backward Extension III

Backward Extendable Condition (Another One)

$\{ S_0, S_1, \cdots \}$ p.d. sequence

$P_k(x)$, $q_k(x)$ 第1種, 第2種 polynomials

次の等式が成立する

$$\sum_{k=0}^{M} q_k(0)^2 - \frac{\left(\sum_0^M p_k(0) q_k(0)\right)^2}{\sum_0^M p_k(0)^2} = - \left| \begin{array}{c|ccc} 0 & S_0 & S_1 \cdots & S_{M-1} \\ \hline S_0 & S_2 & \cdots & S_{M+1} \\ \vdots & \vdots & & \\ S_{M-1} & S_{M+1} & \cdots & S_{2M} \end{array} \right| \Big/ \Delta(S_2 \cdots S_{2M})$$

Proof

$$\sum_0^M p_k(0)^2 = \frac{\Delta(S_2 \cdots S_{2M})}{\Delta(S_0 \cdots S_{2M})}$$

$$\sum_{k=0}^{M} q_k(0)^2 = \frac{-\Delta(0\,0\,S_0 \cdots S_{2M})}{\Delta(S_0 \cdots S_{2M})}$$

$$\sum_{k=0}^{M} p_k(0) q_k(0) = \left| \begin{array}{c|ccc} 0 & S_0 & \cdots & S_{M-1} \\ \hline S_1 & S_2 & \cdots & S_{M+1} \\ \vdots & \vdots & & \\ S_M & S_{M+1} & & S_{2M} \end{array} \right| \Big/ \Delta(S_0 \cdots S_{2M})$$

これを左辺に代入する.

$$\text{左辺} = - \frac{\Delta(0\,0\,S_0 \cdots S_{2M})}{\Delta(S_0 \cdots S_{2M})} - \frac{\left| \begin{array}{c|ccc} 0 & S_0 & \cdots & S_{M-1} \\ \hline S_1 & S_2 & & S_{M+1} \\ \vdots & & \ddots & \\ S_M & S_{M+1} & & S_{2M} \end{array} \right|^2}{\Delta(S_2 \cdots S_{2M})\,\Delta(S_0 \cdots S_{2M})}$$

$$= - \frac{\Delta(S_2 \cdots S_{2M})\,\Delta(0\,0\,S_0 \cdots S_{2M}) + \left| \begin{array}{c|ccc} 0 & S_0 & \cdots & S_{M-1} \\ \hline S_1 & S_2 & \cdots & S_{M+1} \\ \vdots & \vdots & & \\ S_M & S_{M+1} & \cdots & S_{2M} \end{array} \right|^2}{\Delta(S_2 \cdots S_{2M})\,\Delta(S_0 \cdots S_{2M})}$$

分子において, Sylvester型の公式を用いると

$$\Delta(S_2 \cdots S_{2M})\,\Delta(0\,0\,S_0 S_1 \cdots S_{2M}) = \begin{vmatrix} S_2 & \cdots & S_{M+1} \\ \vdots & & \vdots \\ S_{M+1} & \cdots & S_{2M} \end{vmatrix} \begin{vmatrix} 0 & 0 & S_0 & \cdots & S_{M-1} \\ 0 & S_0 & S_1 & \cdots & S_M \\ S_0 & S_1 & S_2 & \cdots & S_{M+1} \\ \vdots & \vdots & \vdots & & \vdots \\ S_{M-1} & S_M & S_{M+1} & \cdots & S_{2M} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & S_0 & \cdots & S_{M-1} \\ S_0 & S_2 & \cdots & S_{M+1} \\ \vdots & \vdots & & \vdots \\ S_{M-1} & S_{M+1} & \cdots & S_{2M} \end{vmatrix} \begin{vmatrix} S_0 & S_1 & \cdots & S_M \\ S_1 & S_2 & \cdots & S_{M+1} \\ \vdots & \vdots & & \vdots \\ S_M & S_{M+1} & \cdots & S_{2M} \end{vmatrix} - \begin{vmatrix} 0 & S_0 & \cdots & S_{M-1} \\ S_1 & S_2 & \cdots & S_{M+1} \\ \vdots & \vdots & & \vdots \\ S_M & S_{M+1} & \cdots & S_{2M} \end{vmatrix}^2$$

であるから

$$\text{左辺} = - \begin{vmatrix} 0 & S_0 & \cdots & S_{M-1} \\ S_0 & S_2 & \cdots & S_{M+1} \\ \vdots & \vdots & & \vdots \\ S_{M-1} & S_{M+1} & \cdots & S_{2M} \end{vmatrix} \Delta(S_0 S_1 \cdots S_{2M}) \Big/ \Delta(S_2 \cdots S_{2M})\,\Delta(S_0 \cdots S_{2M})$$

$$= - \begin{vmatrix} 0 & S_0 & \cdots & S_{M-1} \\ S_0 & S_2 & \cdots & S_{M+1} \\ \vdots & \vdots & & \vdots \\ S_{M-1} & S_{M+1} & \cdots & S_{2M} \end{vmatrix} \Big/ \Delta(S_2 \cdots S_{2M})$$

これは示すべき右辺である。

以上

次に

$\{S_2 S_3 \cdots\}$ に associate した第1種及び第2種多項式を $\tilde{p}_k(x)$, $\tilde{q}_k(x)$ $(k=0,1,2,\ldots)$ とすると、次の等式が成立する。

$$(*)\quad \boxed{ \frac{- \begin{vmatrix} 0 & S_0 & \cdots & S_{M-1} \\ S_0 & S_2 & \cdots & S_{M+1} \\ \vdots & \vdots & & \vdots \\ S_{M-1} & S_{M+1} & \cdots & S_{2M} \end{vmatrix}}{\Delta(S_2 \cdots S_{2M})} = \sum_{k=0}^{M-1} \left(S_0 \tilde{p}_k(0) + S_1 \tilde{p}_k'(0) + \tilde{q}_k'(0) \right)^2 }$$

$$\| $$

$$\sum_{k=0}^{M} q_k(0)^2 - \frac{\left(\sum_{k=0}^{M} p_k(0)\, q_k(0) \right)^2}{\sum_{k=0}^{M} p_k(0)^2}$$

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<u>Lemma 1</u>

$$\left|\begin{array}{c|ccccc} 0 & x & y & S_2 & \cdots & S_{M-1} \\ \hline x & S_2 & \cdots & & & S_{M+1} \\ y & \vdots & & & & \vdots \\ S_2 & \vdots & & & & \vdots \\ \vdots & \vdots & & & & \\ S_{M-1} & S_{M+1} & \cdots & & & S_{2M} \end{array}\right| = Ax^2+By^2+2Cxy+2Dx+2Ey+F$$

とするとき　次が成立している.

(1) $$A = - \begin{vmatrix} S_4 & \cdots & S_{M+2} \\ \vdots & & \vdots \\ S_{M+2} & \cdots & S_{2M} \end{vmatrix} = -\Delta(S_4 \cdots S_{2M})$$

(2) $$B = - \left|\begin{array}{c|cccc} S_2 & S_4 & S_5 & \cdots & S_{M+1} \\ \hline S_4 & S_6 & S_7 & \cdots & S_{M+3} \\ \vdots & \vdots & & & \\ S_{M+1} & S_{M+3} & \cdots & & S_{2M} \end{array}\right|$$

(3) $$C = \left|\begin{array}{c|ccc} S_3 & S_4 & \cdots & S_{M+1} \\ \hline S_5 & S_6 & \cdots & S_{M+3} \\ \vdots & & & \\ S_{M+2} & S_{M+3} & & \end{array}\right|$$

(4) $$D = - \left|\begin{array}{c|cccc} 0 & 0 & S_2 & \cdots & S_{M-1} \\ \hline S_3 & S_4 & \cdots & & S_{M+2} \\ \vdots & \vdots & & & \\ S_{M+1} & S_{M+2} & \cdots & & S_{2M} \end{array}\right|$$

(5) $$E = - \begin{vmatrix} S_2 & \cdots & & & S_{M+1} \\ 0 & 0 & S_2 & \cdots & S_{M-1} \\ S_4 & S_5 & \cdots & & S_{M+3} \\ \vdots & & & & \vdots \\ S_{M+1} & \cdots & & & S_{2M} \end{vmatrix}$$

(6) $$F = + \left|\begin{array}{c|cccc} 0 & 0 & 0 & S_2 \cdots & S_{M-1} \\ \hline 0 & S_2 & \cdots & & S_{M+1} \\ S_2 & \vdots & & & \vdots \\ \vdots & \vdots & & & \\ S_{M-1} & S_{M+1} & \cdots & & S_{2M} \end{array}\right|$$

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Lemma 2

$$\sum_{k=0}^{M} p_k(x)\,p_k(y) = -\frac{1}{\Delta_M}\left|\begin{array}{c|ccc} 0 & 1 & x \cdots & x^M \\ \hline 1 & S_0 & \cdots & S_M \\ y & \vdots & & \\ \vdots & \vdots & & \\ y^M & S_M & \cdots & S_{2M} \end{array}\right|$$ より

(1) $$\sum_{0}^{M} p_k(0)^2 = \frac{\Delta(S_2 \cdots S_{2M})}{\Delta_M}$$

(2) $$\sum_{k=0}^{M} p_k(0)\,p_k'(0) = -\frac{1}{\Delta_M}\left|\begin{array}{c|cccc} 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & S_0 & \cdots & & S_M \\ 1 & \vdots & & & \\ 0 & \vdots & & & \\ \vdots & & & & \\ 0 & S_M & \cdots & & S_{2M} \end{array}\right| = \frac{-1}{\Delta_M}\left|\begin{array}{ccc} S_1 & \cdots & S_M \\ \hline S_3 & \cdots & S_{M+2} \\ \vdots & & \\ S_{M+1} & \cdots & S_{2M} \end{array}\right|$$

(3) $$\sum_{k=0}^{M} p_k'(0)^2 = -\frac{1}{\Delta_M}\left|\begin{array}{c|cccc} 0 & 0 & 1 & 0 \cdots & 0 \\ \hline 0 & S_0 & \cdots & & S_M \\ 1 & \vdots & & & \\ 0 & \vdots & & & \\ \vdots & & & & \\ 0 & S_M & \cdots & & S_{2M} \end{array}\right| = \frac{1}{\Delta_M}\left|\begin{array}{c|ccc} S_0 & S_2 & \cdots & S_M \\ \hline S_2 & S_4 & \cdots & S_{M+2} \\ \vdots & \vdots & & \\ S_M & S_{M+2} & \cdots & S_{2M} \end{array}\right|$$

(4) $$\sum_{k=0}^{M} p_k(0)\,q_k'(0) = \frac{-1}{\Delta_M}\left|\begin{array}{c|cccc} 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & S_0 & \cdots & & S_M \\ 0 & \vdots & & & \\ S_0 & \vdots & & & \\ \vdots & & & & \\ S_{M-2} & S_M & \cdots & & S_{2M} \end{array}\right| = \frac{1}{\Delta_M}\left|\begin{array}{cccc} 0 & S_1 & \cdots & S_M \\ 0 & S_2 & \cdots & S_{M+1} \\ S_0 & \vdots & & \\ \vdots & \vdots & & \\ S_{M-2} & S_{M+1} & \cdots & S_{2M} \end{array}\right|$$

(5) $$\sum_{k=0}^{M} p_k'(0)\,q_k'(0) = \frac{-1}{\Delta_M}\left|\begin{array}{c|ccccc} 0 & 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & S_0 & \cdots & & & S_M \\ 0 & \vdots & & & & \\ S_0 & \vdots & & & & \\ \vdots & & & & & \\ S_{M-2} & S_M & \cdots & & & S_{2M} \end{array}\right| = \frac{1}{\Delta_M}\left|\begin{array}{ccccc} S_0 & 0 & S_2 & \cdots & S_M \\ S_1 & 0 & & & \\ \vdots & S_0 & & & \\ \vdots & \vdots & & & \\ S_M & S_{M-2} & & & S_{2M} \end{array}\right|$$

(6) $$\sum_{k=0}^{M} q_k'(0)^2 = \frac{-1}{\Delta_M}\left|\begin{array}{c|ccccc} 0 & 0 & 0 & S_0 & \cdots & S_{M-2} \\ \hline 0 & S_0 & \cdots & & & S_M \\ 0 & & & & & \\ S_0 & & & & & \\ \vdots & & & & & \\ S_{M-2} & S_M & \cdots & & & S_{2M} \end{array}\right|$$

よって $\{S_2 S_3 \cdots\}$ に associate する polynomials $\hat{p}_k(x)$, $\hat{q}_k(x)$ に対すると,

$(\tilde{1})\quad \sum_{0}^{M-1} \tilde{p}_k(0)^2 = \dfrac{\Delta(S_4 \cdots S_{2M})}{\Delta(S_2 \cdots S_{2M})}$

$(\tilde{2})\quad \sum_{0}^{M-1} \tilde{p}_k(0)\,\tilde{p}_k'(0) = \dfrac{-1}{\Delta(S_2 \cdots S_{2M})} \begin{vmatrix} S_3 & \cdots & S_{M+1} \\ \hline S_5 & \cdots & S_{M+3} \\ \vdots & & \vdots \\ S_{M+2} & \cdots & S_{2M} \end{vmatrix}$

$(\tilde{3})\quad \sum_{0}^{M-1} \tilde{p}_k'(0)^2 = \dfrac{1}{\Delta(S_2 \cdots S_{2M})} \left|\begin{array}{c|ccc} S_2 & S_4 & \cdots & S_{M+1} \\ \hline S_4 & S_6 & \cdots & S_{M+3} \\ \vdots & \vdots & & \\ S_{M+1} & S_{M+3} & \cdots & S_{2M} \end{array}\right|$

$(\tilde{4})\quad \sum_{0}^{M-1} \tilde{p}_k(0)\,\tilde{q}_k'(0) = \dfrac{1}{\Delta(S_2 \cdots S_{2M})} \begin{vmatrix} 0 & S_3 & \cdots & S_{M+1} \\ 0 & S_4 & & \\ S_2 & \vdots & & \\ \vdots & & & \\ S_{M-1} & S_{M+2} & \cdots & S_{2M} \end{vmatrix}$

$(\tilde{5})\quad \sum_{0}^{M-1} \tilde{p}_k'(0)\,\tilde{q}_k'(0) = \dfrac{1}{\Delta(S_2 \cdots S_{2M})} \begin{vmatrix} S_2 & 0 & \cdots & S_{M+1} \\ \vdots & 0 & & \vdots \\ & S_2 & & \\ & \vdots & & \\ S_{M+1} & S_{M-1} & \cdots & S_{2M} \end{vmatrix}$

$(\tilde{6})\quad \sum_{0}^{M-1} \tilde{q}_k'(0)^2 = \dfrac{-1}{\Delta(S_2 \cdots S_{2M})} \left|\begin{array}{c|cccc} 0 & 0 & 0 & S_2 \cdots & S_{M-1} \\ \hline 0 & S_2 & \cdots & & S_{M+1} \\ 0 & \vdots & & & \\ S_2 & & & & \\ \vdots & & & & \\ S_{M-1} & S_{M+1} & \cdots & & S_{2M} \end{array}\right|$

Lemma 1 と Lemma 2 との結果を一緒にすると P.20 の等式(※)を示すことが出来る

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$$(*) \text{の左辺} = \frac{-A}{\Delta(S_2 \cdots S_{2M})} S_0^2 - \frac{B}{\Delta(S_2 \cdots S_{2M})} S_1^2 - \frac{2C}{\Delta(S_2 \cdots S_{2M})} S_0 S_1$$

$$- \frac{2D}{\Delta(S_2 \cdots S_{2M})} S_0 - \frac{2E}{\Delta(S_2 \cdots S_{2M})} S_1 - \frac{F}{\Delta(S_2 \cdots S_{2M})}$$

Lemma1

$$\overset{\downarrow}{=} \frac{\Delta(S_4 \cdots S_{2M})}{\Delta(S_2 \cdots S_{2M})} S_0^2 + \frac{1}{\Delta(S_2 \cdots S_{2M})} \left| \begin{array}{c|ccc} S_2 & S_4 & \cdots & S_{M+1} \\ \hline S_4 & S_6 & \cdots & S_{M+1} \\ \vdots & \vdots & & \\ S_{M+1} & S_{M+3} & \cdots & S_{2M} \end{array} \right| S_1^2$$

$$+ \frac{-2}{\Delta(S_2 \cdots S_{2M})} \left| \begin{array}{c|ccc} S_3 & S_4 & & S_{M+1} \\ \hline S_5 & S_6 & \cdots & S_{M+3} \\ \vdots & & & \\ S_{M+2} & S_{M+3} & \cdots & S_{2M} \end{array} \right| S_0 S_1$$

$$+ \frac{+2}{\Delta(S_2 \cdots S_{2M})} \left| \begin{array}{c|cccc} 0 & 0 & S_2 & \cdots & S_{M-1} \\ \hline S_3 & S_4 & & \cdots & S_{M+2} \\ & & & & \\ S_{M+1} & S_{M+2} & & \cdots & S_{2M} \end{array} \right| S_0$$

$$+ \frac{2}{\Delta(S_2 \cdots S_{2M})} \left| \begin{array}{cccc} S_2 & & \cdots & S_{M+1} \\ 0 & 0 S_2 & \cdots & S_{M+1} \\ S_4 & S_5 & \cdots & S_{M+3} \\ \vdots & & & \\ S_{M+1} & & \cdots & S_{2M} \end{array} \right| S_1$$

$$+ \frac{-1}{\Delta(S_2 \cdots S_{2M})} \left| \begin{array}{c|cccc} 0 & 0 & 0 & S_2 \cdots & S_{M-1} \\ \hline 0 & S_2 & & \cdots & S_{M+1} \\ 0 & \vdots & & & \\ S_2 & \vdots & & & \\ \vdots & S_{M+1} & & \cdots & S_{2M} \\ S_{M-1} & & & & \end{array} \right|$$

Lemma 2

$$\overset{\downarrow}{=}$$

$$= \left(\sum_{k=0}^{M-1} \tilde{p}_k(0)^2\right) S_0^2 + \left(\sum_0^{M-1} \tilde{p}_k'(0)^2\right) S_1^2$$

$$+ 2\left(\sum_0^{M-1} \tilde{p}_k(0)\tilde{p}_k'(0)\right) S_0 S_1$$

$$+ 2\left(\sum_0^{M-1} \tilde{p}_k(0)\tilde{q}_k'(0)\right) S_0$$

$$+ 2\left(\sum_0^{M-1} \tilde{p}_k'(0)\tilde{q}_k'(0)\right) S_1$$

(*) の右辺 ‖

$$+ \sum_0^{M-1} \tilde{q}_k'(0)^2 \qquad = \sum_{k=0}^{M-1} \left(\tilde{p}_k(0) S_0 + \tilde{p}_k'(0) S_1 + \tilde{q}_k'(0)\right)^2$$

よって (*) p. 20 は証明された.

以上

Theorem $\{S_0\, S_1 \cdots\}$ が backward extendable である

ための必要かつ十分条件は

(*) $$D = \sup_M \left[\sum_0^M q_k(0)^2 - \frac{\left(\sum_0^M p_k(0)\, q_k(0)\right)^2}{\sum_0^M p_k(0)^2} \right] < +\infty$$

または,

(**) $$\sum_{k=0}^{\infty} \left(\tilde{p}_k(0) S_0 + \tilde{p}_k'(0) S_1 + \tilde{q}_k'(0)\right)^2 < +\infty$$

さらに,

$$\inf_{\mu \in V(\{S_n\})} \int \frac{1}{x^2}\, d\mu = \sup_M \left[\sum_0^M q_k(0)^2 - \frac{\left(\sum_0^M p_k(0)\, q_k(0)\right)^2}{\sum_0^M p_k(0)^2} \right]$$

$$= \sum_{k=0}^{\infty} \left(\tilde{p}_k(0) S_0 + \tilde{p}_k'(0) S_1 + \tilde{q}_k'(0)\right)^2$$ が成立する

Backward Extension の図表

May 10, 1996

$\{S_n\}$: p.d. 無限列.

$V = V(\{S_n\})$: 表現測度全体., $\{S_n\}$が determinateのときは $V=\{\mu_0\}$

$V_0 = V_0(\{S_n\})$: $\{S_n\}$が indeterminateで, 0-canonical solution の全体.

$p_n(z)$, $q_n(z)$ $(n=0,1,2\cdots)$: 直交多項式, 第2種多項式

$$P = \sum_{n=0}^{\infty} p_n(0)^2,\quad Q = \sum_{n=0}^{\infty} q_n(0)^2,\quad 0 < P, Q \leqq +\infty$$

$$D = \sup_n \left[\sum_{k=0}^{n} q_k(0)^2 - \frac{\left(\sum_{k=0}^{n} p_k(0)\, q_k(0)\right)^2}{\sum_{k=0}^{n} p_k(0)^2} \right],\quad 0 < D \leqq +\infty$$

$$y_\infty = \lim_{n\to\infty} - \frac{\sum_{k=0}^{n} p_k(0)\, q_k(0)}{\sum_{k=0}^{n} p_k(0)^2}$$ （$D<+\infty$のとき lim は存在する）

$$R_\infty = \left\{ (x, y) \;\middle|\; x \geqq \sum_{k=0}^{\infty} \left(p_k(0)\, y + q_k(0) \right)^2 = P(y-y_\infty)^2 + D \right\}$$

ただし $P=\infty$ のときは $R_\infty = \{(x,y) \mid x \geqq D,\ y = y_\infty\}$

<table>
<tr><th>$\{S_n\}$</th><th>P, Q</th><th>D</th><th>Extension</th><th>$V(\{S_n\})$</th><th>$(\alpha,\beta)\in R_\infty$</th><th>$(\alpha,\beta,S_0,S_1,\cdots\cdots)$</th></tr>
<tr><td rowspan="2">indeterminate</td><td rowspan="2">$P<+\infty$
$Q<+\infty$</td><td rowspan="4">$D<+\infty$</td><td rowspan="4">Yes</td><td rowspan="4">$\exists\,\mu\in V$
$\int \frac{1}{x^2}d\mu<+\infty$</td><td>$(\alpha,\beta)\in R_\infty^i$ (内点)
$\exists\,\mu\in V\setminus V_0$
$\begin{cases}\alpha=\int\frac{1}{x^2}d\mu\\ \beta=\int\frac{1}{x}d\mu\end{cases}$</td><td>indeterminate</td></tr>
<tr><td>$(\alpha,\beta)\in\partial R_\infty$ (境界)
$\exists\,\mu\in V_0$
$\begin{cases}\alpha=\int\frac{1}{x^2}d\mu\\ \beta=\int\frac{1}{x}d\mu\end{cases}$</td><td rowspan="3">determinate</td></tr>
<tr><td rowspan="4">determinate</td><td>$P=+\infty$
$Q<+\infty$</td><td rowspan="2">$\alpha=\varepsilon+\int\frac{1}{x^2}d\mu_0$
$\varepsilon\geqq 0$
$\beta=y_\infty=\int\frac{1}{x}d\mu_0$</td></tr>
<tr><td rowspan="2">$P=+\infty$
$Q=+\infty$</td></tr>
<tr><td rowspan="2">$D=+\infty$</td><td rowspan="2">No</td><td>$\int\frac{1}{x^2}d\mu_0=+\infty$
$\mu_0(\{0\})=0$</td><td rowspan="2" colspan="2"></td></tr>
<tr><td>$P<+\infty$
$Q=+\infty$</td><td>$\mu_0(\{0\})>0$</td></tr>
</table>

$$\min_{\mu\in V(\{S_n\})} \left(\int_{-\infty}^{+\infty} \frac{1}{x^2}\, d\mu(x) \right) = \sup_n \left[\sum_{k=0}^{n} q_k(0)^2 - \frac{\left(\sum_{k=0}^{n} p_k(0)\, q_k(0)\right)^2}{\sum_{k=0}^{n} p_k(0)^2} \right]$$