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EVERY NORMAL TOEPLITZ MATRIX IS EITHER OF TYPE I OR OF TYPE II*

TAKASHI ITO†

Abstract. In their 1995 manuscript, Farenick and Lee proved that every normal Toeplitz matrix of order less than or equal to 5 is either of type I or of type II and had left open the case of higher order as a conjecture.

The author of this paper settled the conjecture affirmatively. Almost simultaneously, Farenick and Lee themselves proved the conjecture in a work with Krupnik and Krupnik [*SIAM J. Matrix Anal. Appl.*, 17 (1996), pp. 1037–1043]. Since the idea of proof in the work just mentioned is somewhat different from the one of this paper, the editor has recommended that the author publish in a separate paper.

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1. Introduction: Normal Toeplitz matrices.

Toeplitz operators have been studied in connection with many branches of mathematics. Our concern here is finite dimensional Toeplitz operators, especially the *normality* of them. For the background of the problem and for previously known results related to this problem, see [2].

Denote by $[a_1, a_2, \ldots, a_N; a_{-1}, a_{-2}, \ldots, a_{-N}]$ the Toeplitz matrix T_N of order $N+1$ generated by $\{a_j\ ;\ -N \le j \le N\}$:

$$T = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{-N} \\ a_1 & a_0 & \cdots & a_{-(N-1)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N-1} & a_{N-2} & \cdots & a_{-1} \\ a_N & a_{N-1} & \cdots & a_0 \end{bmatrix}.$$

Here we always assume $a_0 = 0$.

Consider the self-commutator $[T_N, T_N^*] \overset{def}{=} T_N T_N^* - T_N^* T_N \equiv [\alpha_{m,n}]_{m,n=1}^{N+1}$. Then we have

$$\alpha_{m,n} = \sum_{k=1}^{N+1} a_{m-k}\overline{a}_{n-k} - \sum_{k=1}^{N+1} \overline{a}_{-(m-k)} a_{-(n-k)}.$$

It is easy to see from this relation the skew-symmetry of the self-commutator with respect to the second diagonal:

$$\alpha_{m,n} = -\alpha_{N+2-n,N+2-m} \quad (1 \le m, n \le N+1).$$

Therefore a necessary and sufficient condition for T_N to be normal is that

$$\alpha_{m+1,n+1} = \alpha_{m,n} \quad (1 \le m, n \le N).$$

When expressed in terms of a_n's this shows that T_N is normal if and only if

$$(1) \quad a_m\overline{a}_n - \overline{a}_{-m}a_{-n} + \overline{a}_{N+1-m}a_{N+1-n} - a_{-(N+1-m)}\overline{a}_{-(N+1-n)} = 0 \quad (1 \le m, n \le N).$$

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† Department of Mathematics, Musashi Institute of Technology, Tamazutsumi 1, Setagaya-ku, Tokyo 158, Japan (uafuruta@ipc.musashi-tech.ac.jp).

whereas T_N is of type II if and only if

$$\begin{bmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-N} \end{bmatrix} = e^{i\omega} \begin{bmatrix} a_N \\ a_{N-1} \\ \vdots \\ a_1 \end{bmatrix} \quad \text{for some } 0 \le \omega < 2\pi. \tag{5}$$

The following property was observed originally by Farenick and Lee [1]. This property will be used later on.

PROPOSITION 3. *Suppose that a Toeplitz matrix T_N is normal. If*

$$\begin{bmatrix} a_{k_0} \\ a_{-k_0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{N+1-k_0} \\ a_{-(N+1-k_0)} \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for some } 1 \le k_0 \le N, \tag{6}$$

then T_N is of type I.

Proof. Since T_N is normal, by putting $m = k_0$ in (1) we have

$$\bar{a}_{N+1-k_0} a_{N+1-n} = a_{-(N+1-k_0)} \bar{a}_{-(N+1-n)} \quad (1 \le n \le N).$$

By putting $n = k_0$ it follows $|a_{N+1-k_0}| = |a_{-(N+1-k_0)}|$. Then by assumption (6) we have

$$|a_{N+1-k_0}| = |a_{-(N+1-k_0)}| > 0.$$

Thus

$$a_{N+1-n} = e^{i\theta} \bar{a}_{-(N+1-n)} \quad (1 \le n \le N) \quad \text{where} \quad e^{i\theta} = \frac{a_{-(N+1-k_0)}}{\bar{a}_{N+1-k_0}}. \qquad \square$$

Corresponding to Proposition 3 we have the following for type II.

PROPOSITION 4. *Suppose that a Toeplitz matrix T_N is normal. If*

$$\begin{bmatrix} a_{k_0} \\ a_{-(N+1-k_0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{N+1-k_0} \\ a_{-k_0} \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for some} \quad 1 \le k_0 \le N, \tag{7}$$

then T_N is of type II.

Proof. By putting $m = k_0$ in (1) we have

$$-\bar{a}_{-k_0} a_{-n} + \bar{a}_{N+1-k_0} a_{N+1-n} = 0 \quad (1 \le n \le N).$$

By putting $n = k_0$ in, it follows $|a_{k_0}| = |a_{N+1-k_0}|$. By (7) we have

$$|a_{k_0}| = |a_{N+1-k_0}| > 0.$$

Thus

$$a_{-n} = e^{i\omega} a_{N+1-n} \quad (1 \le n \le N) \quad \text{where} \quad e^{i\omega} = \frac{\bar{a}_{N+1-k_0}}{\bar{a}_{-k_0}}. \qquad \square$$

3. Proof of theorems. Farenick and Lee [1] proved first that every normal Toeplitz matrix of order less than or equal to 5 is either of type I or of type II. Based on this result, we will show that the same statement holds for general orders. Since Farenick et al. [2] does not contain explicitly the original argument of Farenick and Lee [1], we first show their result with a simplified proof.

THEOREM 1 (see Farenick and Lee [1]). *For $1 \le N \le 4$ every normal Toeplitz matrix $T_N = [a_1, \ldots, a_N; a_{-1}, \ldots, a_{-N}]$ is either of type* I *or of type* II.

Proof. 1. Case of $N = 1$. T_N is normal if and only if $|a_1| = |a_{-1}|$. Thus T_N is trivially of type I (and also type II).

2. Case of $N = 2$. Normality of T_N implies from (1) that

$$a_1\bar{a}_2 = a_{-2}\bar{a}_{-1} \tag{8}$$

and

$$|a_1|^2 + |a_2|^2 = |a_{-1}|^2 + |a_{-2}|^2. \tag{9}$$

By our assumption (3) and Proposition 3, if

$$\begin{bmatrix} a_1 \\ a_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_2 \\ a_{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then T_N is of type I. Similarly, by (3) and Proposition 4, if

$$\begin{bmatrix} a_1 \\ a_{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_2 \\ a_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then T_N is of type II. Thus we consider only the case in which none of a_1, a_2, a_{-1}, a_{-2} is zero. Then $|a_1\bar{a}_2| = |a_{-2}\bar{a}_{-1}|$ and (9) imply that

$$|a_1| = |a_{-1}| \text{ and } |a_2| = |a_{-2}| \quad \text{or} \quad |a_1| = |a_{-2}| \text{ and } |a_{-1}| = |a_2|.$$

When $|a_1| = |a_{-1}|$ and $|a_2| = |a_{-2}|$ holds, (8) shows

$$\frac{a_1}{\bar{a}_{-1}} = \frac{a_{-2}}{\bar{a}_2} = e^{i\theta} \quad \text{for some } 0 \le \theta < 2\pi;$$

thus T_N is of type I. Similarly, if $|a_1| = |a_{-2}|$ and $|a_{-1}| = |a_2|$ holds, then T_N is of type II.

3. Case of $N = 3$. By Proposition 1, the Toeplitz submatrix of order 3 $[a_1, a_3; a_{-1}, a_{-3}]$ is normal. Thus, the previous case of $N = 2$ shows that

$$\begin{bmatrix} a_{-1} \\ a_{-3} \end{bmatrix} = e^{i\theta} \begin{bmatrix} \bar{a}_1 \\ \bar{a}_3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_{-1} \\ a_{-3} \end{bmatrix} = e^{i\theta} \begin{bmatrix} a_3 \\ a_1 \end{bmatrix} \quad \text{for some } 0 \le \theta < 2\pi. \tag{10}$$

Furthermore (1) implies

$$|a_2| = |a_{-2}| \tag{11}$$

and

$$a_1\bar{a}_2 + a_2\bar{a}_3 = \bar{a}_{-1}a_{-2} + \bar{a}_{-2}a_{-3}. \tag{12}$$

By our assumption (3) it follows that $|a_2| = |a_2| > 0$.

We have to consider two cases separately.

(i) Suppose

$$\begin{bmatrix} a_{-1} \\ a_{-3} \end{bmatrix} = e^{i\theta} \begin{bmatrix} \bar{a}_1 \\ \bar{a}_3 \end{bmatrix}$$

holds. Set $a_{-2} = e^{i\omega}\bar{a}_2$. If $\omega = \theta$, it is clear that T_N is of type I. Assume $\omega \ne \theta$. By substituting

$$a_{-1} = e^{i\theta}\bar{a}_1, a_{-3} = e^{i\theta}\bar{a}_3, \text{ and } a_{-2} = e^{i\omega}\bar{a}_2$$

into (12), we have

$$(1 - e^{i(\omega-\theta)})(a_1\bar{a}_2 - a_{-3}\bar{a}_{-2}) = 0.$$

Suppose that T_N is normal; that is, (1) holds. If a subset $\{j_1, j_2, \dots, j_M\}$ of $\{1, 2, \dots, N\}$ with $j_1 < j_2 < \cdots < j_M$ is closed under the mapping $j \longmapsto N+1-j$, that is,

$$j_{M+1-m} = N+1-j_m \quad (1 \le m \le M),$$

it is seen from (1) that the Toeplitz submatrix $[a_{j_1}, a_{j_2}, \dots, a_{j_M}; a_{-j_1}, a_{-j_2}, \dots, a_{-j_M}]$ becomes normal. Let us denote this Toeplitz matrix by $[j_1, j_2, \dots, j_M]$:

$$[j_1, j_2, \dots, j_M] \overset{def}{=} [a_{j_1}, a_{j_2}, \dots, a_{j_M}; a_{-j_1}, a_{-j_2}, \dots, a_{-j_M}].$$

In particular, the Toeplitz matrices of order 5

$$(2) \qquad [m, n, N+1-n, N+1-m] \quad \left(1 \le m < n < \frac{N+1}{2}\right)$$

and those of order 4 or 3

$$(2') \qquad \left[m, \frac{N+1}{2}, N+1-m\right] \quad \left(1 \le m < \frac{N+1}{2};\ N \text{ odd}\right),$$

$$(2'') \qquad [m, N+1-m] \quad \left(1 \le m < \frac{N+1}{2}\right)$$

are normal.

As an important consequence of (1), we summarize the above-mentioned facts in a proposition.

PROPOSITION 1. *A Toeplitz matrix T_N is normal if and only if all Toeplitz submatrices with order less than or equal to 5 of the form* (2), (2'), *and* (2'') *are normal.*

Another consequence of (1) is the following proposition.

PROPOSITION 2. *If for some $1 \le m \le N$*

$$\begin{bmatrix} a_m & a_{-m} \\ a_{N+1-m} & a_{-(N+1-m)} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

then normality of T_N is equivalent to that of the Toeplitz submatrix generated $\{a_j\ ;\ j \ne m, -m, N+1-m, -(N+1-m)\}$ with a canonical indexing.

Therefore throughout this paper we shall assume

$$(3) \qquad \begin{bmatrix} a_m & a_{-m} \\ a_{N+1-m} & a_{-(N+1-m)} \end{bmatrix} \ne \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (1 \le m \le N).$$

2. Type I and type II. Recall the definitions of types of normal Toeplitz matrices in [1, 2]. A normal Toeplitz matrix $T_N = [a_1, \dots, a_n; a_{-1}, \dots, a_{-N}]$ is said to be of type I if it is of the form $T = \alpha I + \beta H$ for some scalars α and β and for Hermitian Toeplitz matrix H. It is said to be of type II if it is a *generalized circulant* in the sense that there is $0 \le \omega < 2\pi$ such that

$$a_{-j} = e^{i\omega} a_{N+1-j} \quad (j = 1, 2, \dots, N).$$

It is obvious that a normal Toeplitz matrix T_N (with $a_0 = 0$) is of type I if and only if

$$(4) \qquad \begin{bmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-N} \end{bmatrix} = e^{i\theta} \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_N \end{bmatrix} \qquad \text{for some } 0 \le \theta < 2\pi,$$

Hence we have $a_1\bar{a}_2 = a_{-3}\bar{a}_{-2}$. Thus, by setting $a_{-2} = e^{i\gamma}a_2$, we have

$$a_{-3} = \frac{\bar{a}_2}{\bar{a}_{-2}}a_1 = e^{i\gamma}a_1$$

and

$$a_{-1} = e^{i\theta}\bar{a}_1 = e^{i\theta}e^{i\gamma}\bar{a}_{-3} = e^{i\theta}e^{i\gamma}e^{-i\theta}a_3 = e^{i\gamma}a_3.$$

Thus T_N is of type II.

With a similar argument to that in case (i) we can see the following.

(ii) Suppose

$$\begin{bmatrix} a_{-1} \\ a_{-3} \end{bmatrix} = e^{i\theta}\begin{bmatrix} a_3 \\ a_1 \end{bmatrix}$$

holds. Then T_N is either of type I or of type II.

Before going into the case of $N = 4$, we formulate as a lemma the argument which was used just above, because the same argument will be used later in different situations several times.

LEMMA 1.

(i) $\begin{bmatrix} a_{-1} \\ a_{-3} \end{bmatrix} = e^{i\theta}\begin{bmatrix} \bar{a}_1 \\ \bar{a}_3 \end{bmatrix}$, $a_1\bar{c} = a_{-3}\bar{d}$, *and* $|c| = |d| > 0$ *imply*

$$\begin{bmatrix} a_{-1} \\ d \\ a_{-3} \end{bmatrix} = e^{i\gamma}\begin{bmatrix} a_3 \\ c \\ a_1 \end{bmatrix} \quad \text{where} \quad e^{i\gamma} = \frac{d}{c}.$$

(ii) $\begin{bmatrix} a_{-1} \\ a_{-3} \end{bmatrix} = e^{i\theta}\begin{bmatrix} a_3 \\ a_1 \end{bmatrix}$, $\bar{a}_1 d = a_{-1}\bar{c}$, *and* $|c| = |d| > 0$ *imply*

$$\begin{bmatrix} a_{-1} \\ d \\ a_{-3} \end{bmatrix} = e^{i\gamma}\begin{bmatrix} \bar{a}_1 \\ \bar{c} \\ \bar{a}_3 \end{bmatrix} \quad \text{where} \quad e^{i\gamma} = \frac{d}{\bar{c}}.$$

4. Case of $N = 4$. By Proposition 1, the Toeplitz submatrices of order 3, $[a_1, a_4; a_{-1}, a_{-4}]$, and $[a_2, a_3; a_{-2}, a_{-3}]$ are normal. The case of $N = 3$ shows that for some $0 \le \theta, \omega < 2\pi$,

$$\begin{bmatrix} a_{-1} \\ a_{-4} \end{bmatrix} = e^{i\theta}\begin{bmatrix} \bar{a}_1 \\ \bar{a}_4 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_{-1} \\ a_{-4} \end{bmatrix} = e^{i\theta}\begin{bmatrix} a_4 \\ a_1 \end{bmatrix} \tag{13}$$

and

$$\begin{bmatrix} a_{-2} \\ a_{-3} \end{bmatrix} = e^{i\omega}\begin{bmatrix} \bar{a}_2 \\ \bar{a}_3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_{-2} \\ a_{-3} \end{bmatrix} = e^{i\omega}\begin{bmatrix} a_3 \\ a_2 \end{bmatrix}. \tag{14}$$

Furthermore, (1) implies

$$a_1\bar{a}_2 + a_3\bar{a}_4 = a_{-2}\bar{a}_{-1} + a_{-4}\bar{a}_{-3} \tag{15}$$

and

$$a_1\bar{a}_3 + a_2\bar{a}_4 = a_{-3}\bar{a}_{-1} + a_{-4}\bar{a}_{-2}. \tag{16}$$

Here notice one remark. Under conditions (13) and (14), Propositions 3 and 4 show that if one of the eight elements a_i $(-4 \le i \le 4)$ is zero, then T_N is either of type I or of type II. Therefore we assume that none of a_i is zero. There are four (essentially two) cases to be considered.

(i) Suppose

$$\begin{bmatrix} a_{-1} \\ a_{-4} \end{bmatrix} = e^{i\theta} \begin{bmatrix} \overline{a}_1 \\ \overline{a}_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{-2} \\ a_{-3} \end{bmatrix} = e^{i\omega} \begin{bmatrix} \overline{a}_2 \\ \overline{a}_3 \end{bmatrix}. \tag{17}$$

If $\theta = \omega$, then T_N is of type I. Assume $\theta \neq \omega$. By substituting (17) into (15) and (16) we have

$$(1 - e^{i(\omega-\theta)})(a_1\overline{a}_2 - a_{-4}\overline{a}_{-3}) = 0$$

and

$$(1 - e^{i(\omega-\theta)})(a_1\overline{a}_3 - a_{-4}\overline{a}_{-2}) = 0.$$

Since $\theta \neq \omega$, we have $a_1\overline{a}_2 = a_{-4}\overline{a}_{-3}$ and $a_1\overline{a}_3 = a_{-4}\overline{a}_{-2}$. By multiplying these two equalities, we have $a_1a_{-4}\overline{a}_2\overline{a}_{-2} = a_1a_{-4}\overline{a}_3\overline{a}_{-3}$. Thus $a_2a_{-2} = a_3a_{-3}$; hence we have $|a_{-2}| = |a_2| = |a_3| = |a_{-3}|$. By applying Lemma 1, we can see that

$$\begin{bmatrix} a_{-1} \\ a_{-4} \end{bmatrix} = e^{i\theta} \begin{bmatrix} \overline{a}_1 \\ \overline{a}_2 \end{bmatrix},$$

$a_1\overline{a}_3 = a_{-4}\overline{a}_{-2}$, and $|a_2| = |a_{-3}| > 0$ imply

$$\begin{bmatrix} a_{-1} \\ a_{-3} \\ a_{-4} \end{bmatrix} = e^{i\gamma} \begin{bmatrix} a_4 \\ a_2 \\ a_1 \end{bmatrix} \quad \text{where} \quad e^{i\gamma} = \frac{a_{-3}}{a_2}.$$

Similarly by applying Lemma 1 again, we have

$$\begin{bmatrix} a_{-1} \\ a_{-2} \\ a_{-4} \end{bmatrix} = e^{i\delta} \begin{bmatrix} a_4 \\ a_3 \\ a_1 \end{bmatrix} \quad \text{where} \quad e^{i\delta} = \frac{a_{-2}}{a_3}.$$

However it is clear from these two equations that

$$e^{i\gamma} = \frac{a_{-1}}{a_4} = e^{i\delta}.$$

Thus T_N is of type II.

Using arguments similar to those employed in case (i), we can deduce the following.

(ii) Suppose

$$\begin{bmatrix} a_{-1} \\ a_{-4} \end{bmatrix} = e^{i\theta} \begin{bmatrix} a_4 \\ a_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{-2} \\ a_{-3} \end{bmatrix} = e^{i\omega} \begin{bmatrix} a_3 \\ a_2 \end{bmatrix}.$$

Then T_N is either of type I or of type II.

(iii) Suppose

$$\begin{bmatrix} a_{-1} \\ a_{-4} \end{bmatrix} = e^{i\theta} \begin{bmatrix} \overline{a}_1 \\ \overline{a}_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{-2} \\ a_{-3} \end{bmatrix} = e^{i\omega} \begin{bmatrix} a_3 \\ a_2 \end{bmatrix}. \tag{18}$$

By substituting the first equality of (18) into (15) and (16), we have

$$a_1(\overline{a}_2 - e^{-i\theta}a_{-2}) + \overline{a}_4(a_3 - e^{i\theta}\overline{a}_{-3}) = 0 \tag{19}$$

and

$$a_1(\overline{a}_3 - e^{-i\theta}a_{-3}) + \overline{a}_4(a_2 - e^{i\theta}\overline{a}_{-2}) = 0. \tag{20}$$

By multiplying these two, we have

$$a_1\overline{a}_4|a_2 - e^{i\theta}\overline{a}_{-2}|^2 = a_1\overline{a}_4|a_3 - e^{i\theta}\overline{a}_{-3}|^2.$$

Thus we have

$$|a_2 - e^{i\theta}\overline{a}_{-2}| = |a_3 - e^{i\theta}\overline{a}_{-3}|.$$

If

$$|a_2 - e^{i\theta}\overline{a}_{-2}| = |a_3 - e^{i\theta}\overline{a}_{-3}| = 0,$$

it is clear that T_N is of type I. Suppose

$$|a_2 - e^{i\theta}\overline{a}_{-2}| = |a_3 - e^{i\theta}\overline{a}_{-3}| > 0;$$

then we see $|a_1| = |a_4|$ from (19) or (20). Set $\overline{a}_4 = e^{i\alpha}a_1$, and substitute this into (19) and (20). Then we have

$$\overline{a}_2 - e^{-i\theta}a_{-2} + e^{i\alpha}(a_3 - e^{i\theta}\overline{a}_{-3}) = 0$$

and

$$a_3 - e^{i\theta}\overline{a}_{-3} + e^{i\alpha}(\overline{a}_2 - e^{-i\theta}a_{-2}) = 0.$$

By using the second equality of (18), furthermore, we have

$$(1 - e^{i(\alpha+\theta-\omega)})(\overline{a}_2 - e^{i(\omega-\theta)}a_3) = 0.$$

Thus we have two possibilities: $\alpha + \theta - \omega = 0 \pmod{2\pi}$ or $\overline{a}_2 = e^{i(\omega-\theta)}a_3$.

If $\alpha + \theta - \omega = 0 \pmod{2\pi}$, then

$$\overline{a}_4 = e^{i\alpha}a_1 = e^{i(\omega-\theta)}a_1 = e^{i(\omega-\theta)}e^{i\theta}\overline{a}_{-1} = e^{i\omega}\overline{a}_{-1};$$

hence

$$a_{-1} = e^{i\omega}a_4 \text{ and } a_{-4} = e^{i\theta}\overline{a}_4 = e^{i\theta}e^{i(\omega-\theta)}a_1 = e^{i\omega}a_1.$$

Thus T_N is of type II.

If $\overline{a}_2 = e^{i(\omega-\theta)}a_3$, then

$$\overline{a}_2 = e^{i(\omega-\theta)}e^{-i\omega}a_{-2} = e^{-i\theta}a_{-2};$$

hence

$$a_{-2} = e^{i\theta}\overline{a}_2 \text{ and } a_{-3} = e^{i\omega}a_2 = e^{i\omega}e^{-i(\omega-\theta)}\overline{a}_3 = e^{i\theta}\overline{a}_3.$$

Thus T_N is of type I.

(iv) Suppose

$$\begin{bmatrix} a_{-1} \\ a_{-4} \end{bmatrix} = e^{i\theta}\begin{bmatrix} a_4 \\ a_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{-2} \\ a_{-3} \end{bmatrix} = e^{i\omega}\begin{bmatrix} \overline{a}_2 \\ \overline{a}_3 \end{bmatrix}.$$

In this case, by the same argument as in the previous case (iii) we can show that T_N is either of type I or of type II. □

Before going into the general case, let us observe (Toeplitz) submatrices of T_N of order 5.

LEMMA 2. *Suppose that* $1 \le m < n < \frac{N+1}{2}, 1 \le k < l < \frac{N+1}{2}$, *and*

$$\{m,n\} \cap \{k,l\} = \{r\}, \{m,n\} \cup \{k,l\} \setminus \{r\} = \{p,q\}.$$

If both $[m, n, N+1-n, N+1-m]$ *and* $[k, l, N+1-l, N+1-k]$ *are simultaneously of type* I *(or type* II*), so is* $[p, q, N+1-p, N+1-q]$.

Proof. Let us consider only the type II case. Supposing $m = p, q = l, n = k = r$, for instance, we see that both $[p, r, N+1-r, N+1-p]$ and $[r, q, N+1-q, N+1-r]$ are of type II. Then there are $0 \le \theta, \omega < 2\pi$ such that

$$\begin{bmatrix} a_{-p} \\ a_{-r} \\ a_{-(N+1-r)} \\ a_{-(N+1-p)} \end{bmatrix} = e^{i\theta} \begin{bmatrix} a_{N+1-p} \\ a_{N+1-r} \\ a_r \\ a_p \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{-r} \\ a_{-q} \\ a_{-(N+1-q)} \\ a_{-(N+1-r)} \end{bmatrix} = e^{i\omega} \begin{bmatrix} a_{N+1-r} \\ a_{N+1-q} \\ a_q \\ a_r \end{bmatrix},$$

so that

$$a_{-r} = e^{i\theta} a_{N+1-r} \quad \text{and} \quad a_{-(N+1-r)} = e^{i\theta} a_r$$

and

$$a_{-r} = e^{i\omega} a_{N+1-r} \quad \text{and} \quad a_{-(N+1-r)} = e^{i\omega} a_r.$$

Since

$$\begin{bmatrix} a_r & a_{-r} \\ a_{N+1-r} & a_{-(N+1-r)} \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

by assumption (3), we can conclude $\theta = \omega$. Therefore we have

$$\begin{bmatrix} a_{-p} \\ a_{-q} \\ a_{-(N+1-q)} \\ a_{-(N+1-p)} \end{bmatrix} = e^{i\theta} \begin{bmatrix} a_{N+1-p} \\ a_{N+1-q} \\ a_q \\ a_p \end{bmatrix},$$

and consequently $[p, q, N+1-q, N+1-p]$ is of type II. □

THEOREM 2. *Every normal Toeplitz matrix* $T_N = [a_1, \ldots, a_N; a_{-1}, \ldots, a_{-N}]$ *with* $N \ge 5$ *is either of type* I *or of type* II.

Proof. (a) Case of even N. Since all submatrices of order 5 $[1, k, N+1-k, N]$ $(1 < k < \frac{N+1}{2})$ are normal, by Theorem 1 it is of type I or type II. We claim that they are of the same type. To prove this by contradiction, without loss of generality we may assume that there are k_0, k_1 such that $1 < k_0 < k_1 < \frac{N+1}{2}$ and $[1, k_0, N+1-k_0, N]$ is of type I but not of type II while $[1, k_1, N+1-k_1, N]$ is of type II but not of type I.

Again by Theorem 1 $[k_0, k_1, N+1-k_1, N+1-k_0]$ is either of type I or of type II. If it is of type I, by applying Lemma 2 to $[1, k_0, N+1-k_0, N]$ and $[k_0, k_1, N+1-k_1, N+1-k_0]$, we see that $[1, k_1, N+1-k_1, N]$ is of type I, which is a contradiction. In the same way, assuming type II of $[k_0, k_1, N+1-k_1, N+1-k_0]$ leads to a contradiction again. Thus we have proved that all $[1, k, N+1-k, N]$ $(1 < k < \frac{N+1}{2})$ are of the same type. Suppose, for instance, they are of type I. Then for $1 < k < m < \frac{N+1}{2}$

$$\begin{bmatrix} a_{-1} \\ a_{-k} \\ a_{-(N+1-k)} \\ a_{-N} \end{bmatrix} = e^{i\theta} \begin{bmatrix} \overline{a}_1 \\ \overline{a}_k \\ \overline{a}_{N+1-k} \\ \overline{a}_N \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{-1} \\ a_{-m} \\ a_{-(N+1-m)} \\ a_{-N} \end{bmatrix} = e^{i\omega} \begin{bmatrix} \overline{a}_1 \\ \overline{a}_m \\ \overline{a}_{N+1-m} \\ \overline{a}_N \end{bmatrix}.$$

Since

$$\begin{bmatrix} a_1 & a_{-1} \\ a_N & a_{-N} \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

by assumption (3), this implies $\theta = \omega$, and consequently

$$a_{-k} = e^{i\theta} \overline{a}_k \quad (1 \le k \le N).$$

This proves that T_N is of type I.

In a similar way we can treat the case of type II.

(b) Case of odd N. Since all submatrices of order 5 $[1, k, N+1-k, N]$ $(1 < k < \frac{N+1}{2})$ and the submatrix of order 4 $[1, \frac{N+1}{2}, N]$ are normal, by Theorem 1 each of them is either of type I or of type II. It is shown in case (a) that all such submatrices of order 5 are of the same type. Suppose that they are of type I. In this case we claim that the submatrix $[1, \frac{N+1}{2}, N]$ is of type I too. Suppose, by contradiction, that $[1, \frac{N+1}{2}, N]$ is of type II but not of type I, while for some $1 < k_1 < N$ the submatrix $[1, k_1, N+1-k_1, N]$ is of type I but not of type II. Consider $[k_1, \frac{N+1}{2}, N+1-k_1]$. This is either of type I or of type II by Theorem 1. Suppose that this is of type I. By applying the same idea as in Lemma 2 to the two submatrices $[k_1, \frac{N+1}{2}, N+1-k_1]$ and $[1, k_1, N+1-k_1, N]$, we can see that $[1, \frac{N+1}{2}, N]$ is of type I, which is a contradiction. Suppose that $[k_1, \frac{N+1}{2}, N+1-k_1]$ is of type II. Then, by applying the same idea as in Lemma 2 again to $[k_1, \frac{N+1}{2}, N+1-k_1]$ and $[1, \frac{N+1}{2}, N]$, we see that $[1, k_1, N+1-k_1, N]$ is of type II, which is also a contradiction. Thus we have that all the submatrices of order 5 $[1, k, N+1-k, N]$ $(1 < k < \frac{N+1}{2})$ and the submatrix of order 4 $[1, \frac{N+1}{2}, N]$ are of the same type, say type I. Then we can see, as in case (a), that T_N itself is of type I. $\square$

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